

QCD Sum-Rule Invisibilty of the σ Meson

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Abstract

QCD Laplace sum-rules for light-quark $I = 0, 1$ scalar currents are used to investigate candidates for the lightest $q\bar{q}$ scalar mesons. The theoretical predictions for the sum-rules include instanton contributions which split the degeneracy between the $I = 0$ and $I = 1$ channels. The self-consistency of the theoretical predictions is verified through a Hölder inequality analysis, confirming the existence of an effective instanton contribution to the continuum. The sum-rule analysis indicates that the $f_0(980)$ and $a_0(1450)$ should be interpreted as the lightest $q\bar{q}$ scalar mesons. This apparent decoupling of the $f_0(400 - 1200)$ (or σ) and $a_0(980)$ from the quark scalar currents suggests a non- $q\bar{q}$ interpretation of these resonances.

1 Field-Theoretical Content of the Sum-Rule

The nature of the scalar mesons is a challenging problem in hadronic physics. In particular, a variety of interpretations exist for the lowest-lying isoscalar resonances [$f_0(400 - 1200)$, $f_0(980)$, $f_0(1370)$, $f_0(1500)$] and isovector resonances [$a_0(980)$, $a_0(1450)$] listed by the Particle Data Group (PDG) [1]. In particular, interpreting the $f_0(400 - 1200)$ (or σ) is particularly significant because of its possible interpretation as the σ meson of chiral symmetry breaking. In this paper we will summarize and extend previous work [2] which used QCD Laplace sum-rules to study the various possibilities for the lowest-lying, non-strange quark scalar mesons.

QCD sum-rules probe hadronic properties through correlation functions of appropriately chosen currents. In the $SU(2)$ flavour limit $m_u = m_d \equiv m$, the non-strange-quark $I = 0, 1$ scalar mesons are studied via the scalar-current correlation function:

$$J_I(x) = \frac{m}{2} \left[\bar{u}(x)u(x) + (-1)^I \bar{d}(x)d(x) \right] \quad , \quad I = 0, 1 \quad (1)$$

$$\Pi_I(Q^2) = i \int d^4x e^{iq \cdot x} \langle O | T J_I(x) J_I(0) | O \rangle \quad (2)$$

Laplace sum-rules, which exponentially suppress the high-energy region, are obtained by applying the Borel transform operator \hat{B} to the appropriately-subtracted dispersion relation satisfied by (2) [3]:

$$\mathcal{R}_0^I(\tau) \equiv \frac{1}{\tau} \hat{B} [\Pi_I(Q^2)] = \frac{1}{\pi} \int_0^\infty \text{Im} \Pi_I(t) e^{-t\tau} dt \quad (3)$$

To leading order in the quark mass, the theoretical prediction for \mathcal{R}_0^I incorporates two-loop $\overline{\text{MS}}$ scheme perturbative corrections [4], infinite correlation-length non-perturbative vacuum effects parametrized by the QCD condensates [3, 5], and finite-correlation length non-perturbative effects of instantons in the instanton liquid model [6, 7]:

$$\begin{aligned} \mathcal{R}_0^I(\tau) = & \frac{3m^2}{16\pi^2\tau^2} \left(1 + 4.821098 \frac{\alpha}{\pi} \right) + m^2 \left(\frac{3}{2} \langle m\bar{q}q \rangle + \frac{1}{16\pi} \langle \alpha G^2 \rangle + \pi \langle \mathcal{O}_6 \rangle \tau \right) \\ & + (-1)^I m^2 \frac{3\rho_c^2}{16\pi^2\tau^3} e^{-\frac{\rho_c^2}{2\tau}} \left[K_0 \left(\frac{\rho_c^2}{2\tau} \right) + K_1 \left(\frac{\rho_c^2}{2\tau} \right) \right] \end{aligned} \quad (4)$$

where the quantity $\rho = 1/(600 \text{ MeV})$ is the mean instanton size in the instanton liquid model [7]. The only theoretical source of isospin-breaking effects in (4) are instantons, which are known to have non-trivial contributions for only the scalar and pseudoscalar correlation functions.

We have used $SU(2)$ symmetry for the dimension-four quark condensate contributions to (4) (*i.e.* $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle \equiv \langle \bar{q}q \rangle$). The quantity $\langle \mathcal{O}_6 \rangle$ denotes the dimension-six quark condensates for which the vacuum saturation hypothesis [3] provides a reference value

$$\langle \mathcal{O}_6 \rangle = -f_{vs} \frac{88}{27} \alpha \langle \bar{q}q\bar{q}q \rangle = -f_{vs} 5.9 \times 10^{-4} \text{ GeV}^6 \quad (5)$$

where $f_{vs} = 1$ for exact vacuum saturation. Larger values of effective dimension-six operators found in [8] imply that f_{vs} could be as large as 2, suggesting a central value $f_{vs} = 1.5$. The quark condensate is determined by the GMOR (PCAC) relation, and the gluon condensate is given by [8]

$$\langle \alpha G^2 \rangle = (0.045 \pm 0.014) \text{ GeV}^4 \quad (6)$$

Renormalization group improvement of (4) implies that α and m are running quantities evaluated at the mass scale $Q = \frac{1}{\sqrt{\tau}}$ in the $\overline{\text{MS}}$ scheme. We use $\Lambda_{\overline{\text{MS}}} \approx 300 \text{ MeV}$ for three active flavours, consistent with current estimates of $\alpha(M_\tau)$ and matching conditions through the charm threshold [1, 9].

Phenomenological analysis of the sum-rule (3) proceeds through the resonance plus continuum model [3]

$$\text{Im} \Pi_I(t) = \text{Im} \Pi_I^{\text{res}} + \theta(t - s_0) \text{Im} \Pi_I^{\text{QCD}}(t) \quad (7)$$

where $Im\Pi_I^{res}$ denotes the resonance contributions, and $Im\Pi_I^{QCD}$ represents the theoretically-determined QCD continuum occurring above the continuum threshold s_0 . Defining these continuum contributions as

$$c_0^I(\tau, s_0) = \frac{1}{\pi} \int_{s_0}^{\infty} Im\Pi_I^{QCD}(t) e^{-t\tau} dt \quad (8)$$

leads to a revised sum-rule which isolates the theoretical and phenomenological (resonance) contributions:

$$\mathcal{S}_0^I(\tau, s_0) \equiv \mathcal{R}_0^I(\tau) - c_0^I(\tau, s_0) = \frac{1}{\pi} \int_0^{s_0} Im\Pi_I^{res}(t) e^{-t\tau} dt \quad (9)$$

Traditionally, only the perturbative contributions are included in the continuum. However, the Q^2 analytic structure of the instanton contributions to $\Pi_I^{inst}(Q^2)$ implies the existence of an imaginary part $Im\Pi_I^{inst}(t)$ which leads to the following instanton continuum contribution [10]:

$$c_{0_{inst}}^I(\tau, s_0) = \frac{1}{\pi} \int_{s_0}^{\infty} Im\Pi_I^{inst}(t) e^{-t\tau} dt = (-1)^{I+1} \frac{3m^2}{8\pi} \int_{s_0}^{\infty} t J_1(\rho_c \sqrt{t}) Y_1(\rho_c \sqrt{t}) dt \quad (10)$$

where $J_n(x)$ and $Y_n(x)$ denote Bessel functions. The instanton continuum contribution has been ignored in previous applications of instanton effects in sum-rules. It should be noted that this formulation of the instanton effects leads to improved IR behaviour when integrating over the instanton density because (10) approaches zero in the limit $\rho \rightarrow 0$.

2 Hölder Inequality Constraints

In the phenomenological analysis of QCD sum-rules, the behaviour of $\mathcal{S}_0(\tau, s_0)$ as a function of Borel-parameter τ is used to extract the phenomenological resonance parameters through (9), raising the difficult question of the τ region where the theoretical prediction $\mathcal{S}_0(\tau, s_0)$ is valid [3]. This question can be addressed via Hölder inequalities, which must be upheld if Laplace sum-rules are to be consistent with the physically-required positivity of $Im\Pi_I^{res}(t)$ within the integrand of (9) [11]:

$$\frac{\mathcal{S}_0^I[\omega\tau + (1-\omega)\delta\tau, s_0]}{(\mathcal{S}_0^I[\tau, s_0])^\omega (\mathcal{S}_0^I[\tau + \delta\tau, s_0])^{1-\omega}} \leq 1 \quad , \quad \forall 0 \leq \omega \leq 1 \quad (11)$$

Provided that $\delta\tau$ is reasonably small ($\delta\tau \approx 0.1 \text{ GeV}^{-2}$ appears to be sufficient [11]), these inequalities are insensitive to the choice of $\delta\tau$, permitting a simple analysis of the inequality as a function of the Borel-parameter τ .

The scalar-channel sum-rules satisfy the inequality in a fashion qualitatively similar to other channels [11], supporting the self-consistency of the theoretical predictions. The instanton continuum (10) is crucial to this agreement. Regions of validity in which the sum-rules satisfy the inequality (11) are

$$0.3 \text{ GeV}^{-2} \leq \tau \leq 1.7 \text{ GeV}^{-2} \quad , \quad s_0 > 3 \text{ GeV}^2 \quad (I = 0) \quad (12)$$

$$0.3 \text{ GeV}^{-2} \leq \tau \leq 1.1 \text{ GeV}^{-2} \quad , \quad s_0 > 3 \text{ GeV}^2 \quad (I = 1) \quad (13)$$

3 Phenomenological Analysis

The sum-rule predictions of the properties of the lowest-lying $I = 0, 1$ quark scalar resonances can now be studied through (9). Since the resonances could have a substantial width, it is necessary to extend the narrow width approximation traditionally used in sum-rules. A flexible and numerically simple technique is to build up the resonance shape using n unit-area square pulses [2, 12]

$$\frac{1}{\pi} \text{Im}\Pi^{(n)}(t) = \frac{2}{n\pi} \sum_{j=1}^n \sqrt{\frac{n-j+f}{j-f}} P_M \left[t, \sqrt{\frac{n-j+f}{j-f}} \Gamma \right] \quad (14)$$

$$P_M(t, \Gamma) = \frac{1}{2M\Gamma} \left[\Theta(t - M^2 + M\Gamma) - \Theta(t - M^2 - M\Gamma) \right] \quad (15)$$

A single square pulse models a broad nearly structureless contribution (such as a broad light σ) to $\text{Im}\Pi(t)$, while a Breit-Wigner resonance of a particle of mass M and width Γ can be expressed as a sum of several square pulses. The quantity f can be fixed by normalizing the area of the n -pulse approximation to unity.

We begin the phenomenological analysis with the 4-pulse approximation (14) to $\text{Im}\Pi^{res}(t)$ so that (3) becomes

$$\frac{1}{\pi} \text{Im}\Pi_I^{res} = F^2 M^4 \frac{1}{\pi} \text{Im}\Pi^{(4)}(t) \quad , \quad \mathcal{S}^I(\tau, s_0) = F^2 M^4 e^{-M^2 \tau} W_4(M, \Gamma, \tau) \quad (16)$$

$$W_4(M, \Gamma, \tau) = \frac{2}{4\pi} \sum_{j=1}^4 \frac{1}{M\Gamma\tau} \sinh \left[M \sqrt{\frac{4-j+f}{j-f}} \Gamma \tau \right] \quad (17)$$

where F is the strength with which the scalar current couples the vacuum to the resonance. The free parameters in this expression, the resonance-related quantities F , M , Γ and the continuum-threshold s_0 , can be extracted from a fit to the τ dependence of the theoretical expression $\mathcal{S}^I(\tau, s_0)$. This is done by minimizing the χ^2 defined by

$$\chi^2 = \frac{1}{N} \sum_{j=1}^N \frac{\left[\mathcal{S}^I(\tau_j, s_0) - F^2 M^4 e^{-M^2 \tau_j} W_4(M, \Gamma, \tau_j) \right]^2}{\epsilon(\tau_j)^2} \quad (18)$$

where the sum is over evenly spaced, discrete τ points in the ranges (12,13) consistent with the Hölder inequality. The weighting factor ϵ used for the evaluation of (18) is $\epsilon(\tau) = 0.2 \mathcal{S}^I(\tau, s_0)$. This 20% uncertainty has the desired property of being dominated by the continuum at low τ and power-law corrections at large τ . Other choices of the 0.2 prefactor in ϵ would simply rescale the χ^2 , so its choice has no effect on the values of the χ^2 -minimizing parameters.

In the χ^2 minimization, the quark mass parameter \hat{m} is now absorbed into the quantity $a = F^2 M^4 / \hat{m}^2$. The best-fit parameters are subjected to a Monte-Carlo simulation which includes the parameter ranges $1 \leq f_{vs} \leq 2$, a 15% variation in the instanton size ρ , and a simulation of continuum and OPE truncation uncertainties. This results in the 90% confidence level results for the best-fit parameters shown in Table 1. Decreasing the number of pulses (to simulate a structureless resonance) does not alter the χ^2 , and

I	M (GeV)	s_0 (GeV ²)	a (GeV ⁴)	Γ (GeV)
0	1.00 ± 0.09	3.7 ± 0.4	0.08 ± 0.02	0.19 ± 0.14
1	1.55 ± 0.11	5.0 ± 0.7	0.17 ± 0.04	0.22 ± 0.11

Table 1: *Results of the Monte-Carlo simulation of 90% confidence-level uncertainties for the resonance parameters and continuum threshold for the $I = 0, 1$ channels.*

only leads to a rescaling of Γ . Two-resonance models recover the single-resonance results in Table 1, so there is no evidence of a hidden light resonance in either of the channels.

Thus we conclude that a QCD sum-rule analysis is consistent with the interpretation of the $f_0(980)$ and $a_0(1450)$ as the lightest non-strange quark scalar mesons. A light σ meson [$f_0(400 - 1200)$] and the $a_0(980)$ appear to be decoupled from the quark scalar currents, suggesting a non- $q\bar{q}$ interpretation of these resonances.

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